Metric-Like Spaces as Enriched Categories

Simon Willerton

University of Sheffield

CATMI 2023

Bergen

Enriched Categories

A category $\mathcal{C}$: collection ob$\mathcal{C}$

- $\forall c, c'$ $\quad$ $\mathcal{C}(c,c')$ a set
- $\forall c, c', c''$ $\quad$ $\mathcal{C}(c,c') \times \mathcal{C}(c',c'') \longrightarrow \mathcal{C}(c,c'')$ $\quad$ a function
- $\forall c$ $\quad$ $\mathrm{id} \in \mathcal{C}(c,c)$

+ axioms

For some categories the hom-sets can be considered as 'things' other than sets.

Eg Vect(V,W) is naturally a vector space & composition is bilinear.

The right notion of 'thing' is 'object in a monoidal category $(\mathcal{V}, \otimes, \mathbb{I})$'.

Eg $(\mathrm{Set}, \times, \{*\})$, $(\mathrm{Vect}, \otimes, \mathbb{F})$

ENRICHED CATEGORIES

A category $\mathcal{C}$: collection $\mathrm{ob}\,\mathcal{C}$

- $\forall c, c'$, $\quad \mathcal{C}(c,c') \in \mathrm{Ob}\,\mathrm{Set}$
- $\forall c, c', c''$, $\quad \mathcal{C}(c,c') \times \mathcal{C}(c',c'') \longrightarrow \mathcal{C}(c,c'')$ a morphism in Set
- $\forall c$, $\quad \{*\} \longrightarrow \mathcal{C}(c,c)$ a morphism in Set

+ axioms

For some categories the hom-sets can be considered as 'things' other than sets.

Eg $\mathrm{Vect}(V,W)$ is naturally a vector space & composition is bilinear.

The right notion of 'thing' is 'object in a monoidal category $(\mathcal{V}, \otimes, \mathbb{I})$'.

Eg $(\mathrm{Set}, \times, \{*\})$, $(\mathrm{Vect}, \otimes, \mathbb{F})$

Enriched Categories

A $\mathcal{V}$-category $\mathcal{C}$: collection ob$\mathcal{C}$

- $\forall c, c'$, $\qquad \mathcal{C}(c,c') \in \mathrm{Ob}\,\mathcal{V}$
- $\forall c, c', c''$, $\qquad \mathcal{C}(c,c') \otimes \mathcal{C}(c',c'') \longrightarrow \mathcal{C}(c,c'')$ a morphism in $\mathcal{V}$
- $\forall c$, $\qquad \mathbb{1} \longrightarrow \mathcal{C}(c,c)$ a morphism in $\mathcal{V}$

+ axioms

For some categories the hom-sets can be considered as 'things' other than sets.

Eg Vect(V,W) is naturally a vector space & composition is bilinear.

The right notion of 'thing' is 'object in a monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$'.

Eg $(\mathrm{Set}, \times, \{*\})$, $(\mathrm{Vect}, \otimes, \mathbb{F})$

METRIC SPACES AS ENRICHED CATEGORIES

Do not need $\mathcal{V}$ to be "concrete".

Eg $(\overline{\mathbb{R}}_+, +, 0)$ $\quad$ $\overline{\mathbb{R}}_+$ - objects: $[0,\infty]$

morphisms: $a \to b \iff a \geq b$

$\overline{\mathbb{R}}_+$-category X: collection ob$X$

- $\forall x, x'$, $\quad X(x,x') \in [0,\infty]$
- $\forall x, x', x''$, $\quad X(x,x') + X(x',x'') \geq X(x,x'')$
- $\forall x$ $\quad 0 \geq X(x,x)$

+ <u>no</u> axioms

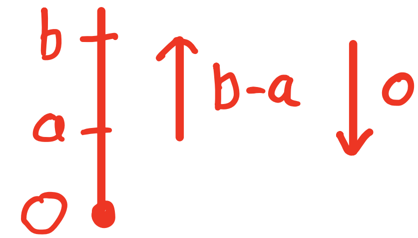
Generalization of classical (Fréchet) metric space.

- Not symmetric
- Allow ∞ distance
- $X(x,x')=0 \not\Rightarrow x=x'$

EXAMPLES OF $\overline{\mathbb{R}}_+$-CATEGORIES

(i) $\overline{\mathbb{R}}_+$: $\overline{\mathbb{R}}_+(a,b) = \max(b-a, 0)$

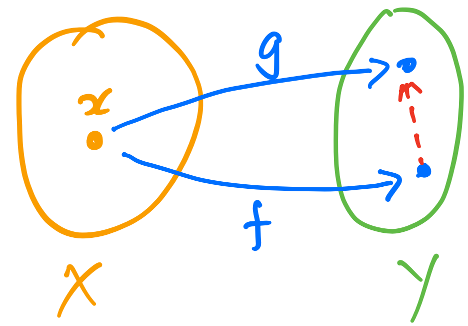
$=: b \dot{-} a$



(ii) $[\![X,Y]\!]$: objects: $\{\overline{\mathbb{R}}_+\text{-functors } X \to Y\}$

distance non-increasing $= \{f : X \to Y \mid X(x,x') \geq Y(f(x), f(x'))\}$

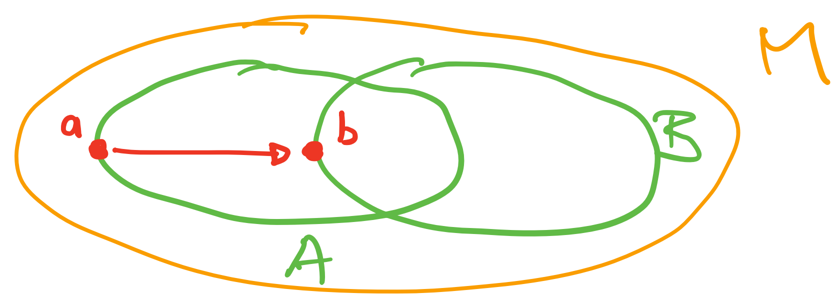
$$[\![X,Y]\!](f,g) := \sup_x Y(f(x), g(x))$$



(iii) M classical metric space $\quad ob(S_M) := \{\text{compact subsets of } M\}$

$$S_M(A,B) = \sup_{a\in A} \inf_{b \in B} M(a,b)$$

$S_M(A,B) = 0 \iff A \subseteq B$



SCALAR-VALUED FUNCTORS AND THE YONEDA EMBEDDING

Should think of $\mathcal{V}$ as the "base category" or "category of scalars".

For "good" $\mathcal{V}$ have $\mathcal{V}$-categories of "scalar-valued functors"

$$[\![\mathcal{C},\mathcal{V}]\!] \quad \& \quad [\![\mathcal{C}^{op},\mathcal{V}]\!]$$

with Yoneda embeddings

$$\mathcal{C} \hookrightarrow [\![\mathcal{C}^{op},\mathcal{V}]\!]; \quad c \mapsto \mathcal{C}(-,c)$$

Analogue of "delta function"

$$S \hookrightarrow \mathrm{Fun}(S,\mathbb{F}); \quad s \mapsto \delta_s$$

Eg. For $\overline{\mathbb{R}}_+$, $\quad \mathrm{ob}[\![X,\overline{\mathbb{R}}_+]\!] := \{f : X \to [0,\infty] \mid X(x,x') \geq f(x') \dot{-} f(x)\}$

$$[\![X,\overline{\mathbb{R}}_+]\!](f,g) := \sup_{x\in X}\{g(x) \dot{-} f(x)\}$$

$$X \hookrightarrow [\![X^{op},\overline{\mathbb{R}}_+]\!]; \quad x \mapsto X(-,x) \qquad \text{isometry}$$

For X classical, this is the "Kuratowski embedding".

NUCLEUS OF A PROFUNCTOR

Given a "matrix"

$$M : R \times S \to \mathbb{F}$$

finite sets

$\mathbb{F}^R = \mathrm{Fun}(R, \mathbb{F})$

get linear maps:

$$\begin{tikzcd} \mathbb{F}^R \arrow[r, bend left, "\bar{M}^T"] & \mathbb{F}^S \arrow[l, bend left, "\bar{M}"] \end{tikzcd}$$

$$(\bar{M} w)(r) := \sum_s M(r,s) w(s)$$

These are adjoint:

$$\langle \bar{M}^T v, w \rangle_{\mathbb{F}^S} = \langle v, \bar{M} w \rangle_{\mathbb{F}^R}$$

Given a "profunctor"

$$P : \mathcal{C}^{op} \otimes \mathcal{D} \to \mathcal{V}$$

$(P : \mathcal{C} \nrightarrow \mathcal{D})$

get an adjunction

"Isbell"

$$\begin{tikzcd} {[\![\mathcal{C}^{op}, \mathcal{V}]\!]} \arrow[r, bend left, "P^*"] \arrow[r, phantom, "\perp"] & {[\![\mathcal{D}, \mathcal{V}]\!]^{op}} \arrow[l, bend left, "P_*"] \end{tikzcd}$$

$$(P_* \psi)(c) := \int_d [P(c,d), \psi(d)]$$

$$\mathrm{Nuc}(P) := \{(\varphi, \psi) \mid P^* \varphi = \psi,\ P_* \psi = \varphi\} \simeq \mathrm{Fix}(P_* P^*) \simeq \mathrm{Fix}(P^* P_*)$$

Tight Span I

M classical metric space, $T(M) \subseteq \mathrm{Fun}(X, \mathbb{R})$ metric space

$M \hookrightarrow T(M)$ via Kuratowski

Injective envelope, Isbell hull, tight span, hyperconvex hull, ...

Minimal convex space with nice properties.

Eg

Has been discovered many times. Many applications.

Eg
- Recreating phylogenetic trees
- Placing servers on a network.

Tight Span II

X generalized metric space. "Hom profunctor":

$$X(-,-) : X^{op} \otimes X \to \overline{\mathbb{R}}_+$$

Get adjunction

$$[\![X^{op}, \overline{\mathbb{R}}_+]\!] \rightleftarrows [\![X, \overline{\mathbb{R}}_+]\!]^{op}$$

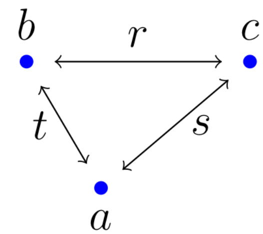
$$\hookleftarrow X \hookrightarrow$$

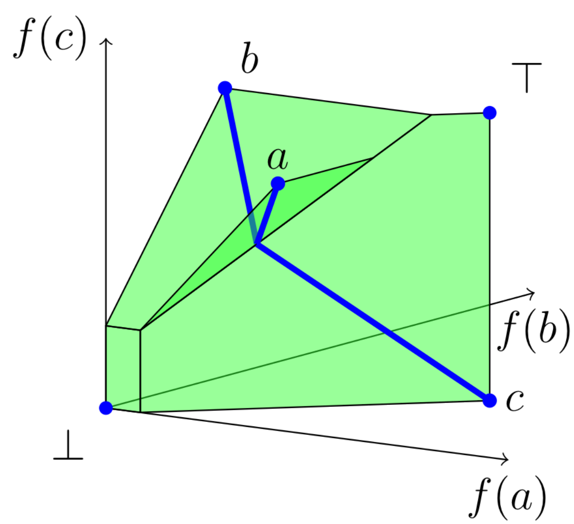
Nucleus is the "Isbell completion" $X \hookrightarrow I(X)$

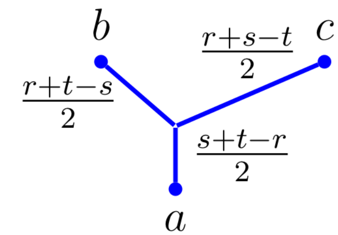
For a classical metric space

$$T(M) \subseteq I(M)$$

largest symmetric subspace







$I(X)$ - "directed tight span"

- Kemajou - Künzi - Olela Otafadu
- Hirai - Koichi (multicommodity flow)

Notion of Euler characteristic

finite $\mathcal{V}$-cats ($\mathcal{V}$ with size) — "magnitude"

finite cats

Leinster

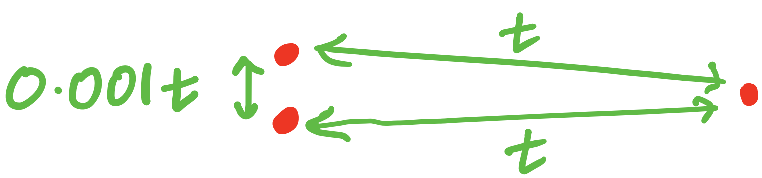
finite metric spaces

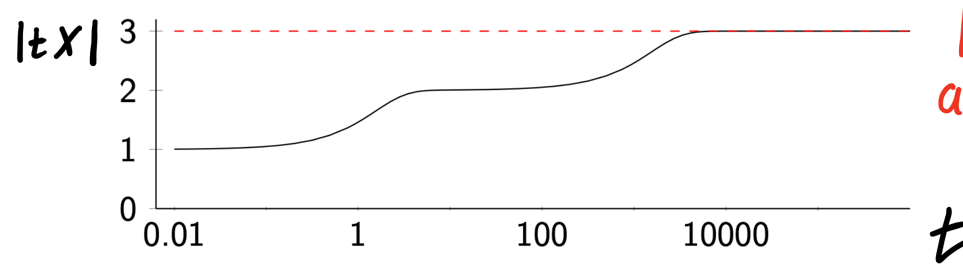
finite sets

finite posets [Rota]

finite groups [Wall]

X finite metric space, magnitude $|X| \in \mathbb{R}$ (if it exists)





$|tX| \to \#X$ as $t \to \infty$

Magnitude measures "effective number of points".

Magnitude

Volume etc of subsets of $\mathbb{R}^n$

(Bio)diversity

Minkowski dimension

Potential theory & special functions

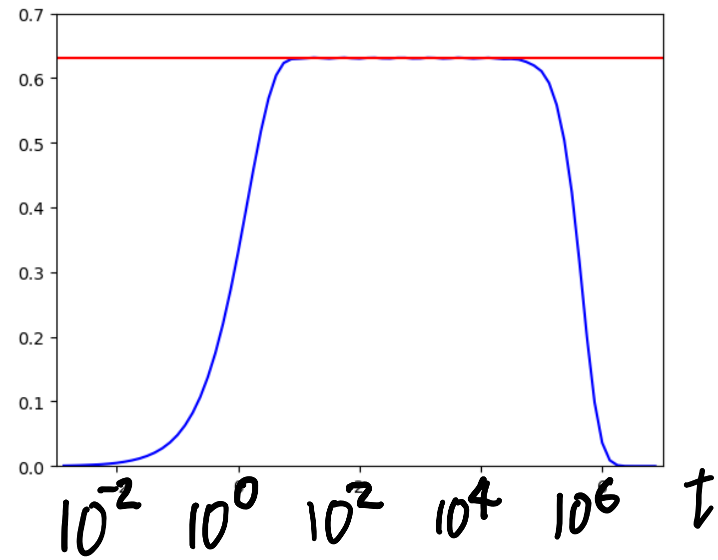
Magnitude homology

MAGNITUDE III

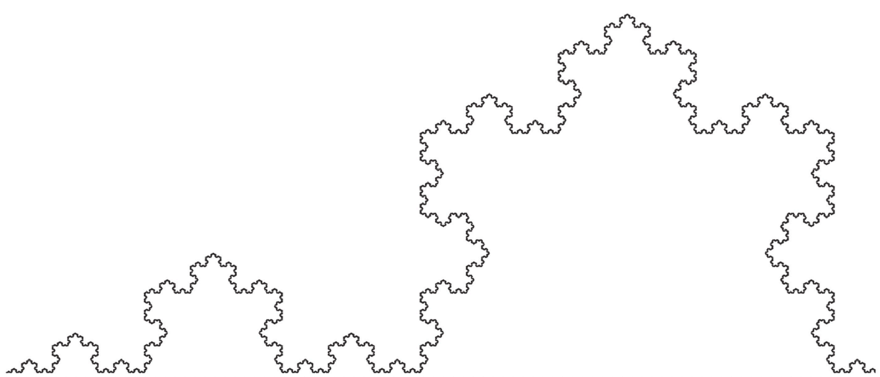
10A

Cantor set, C, with 4096 points

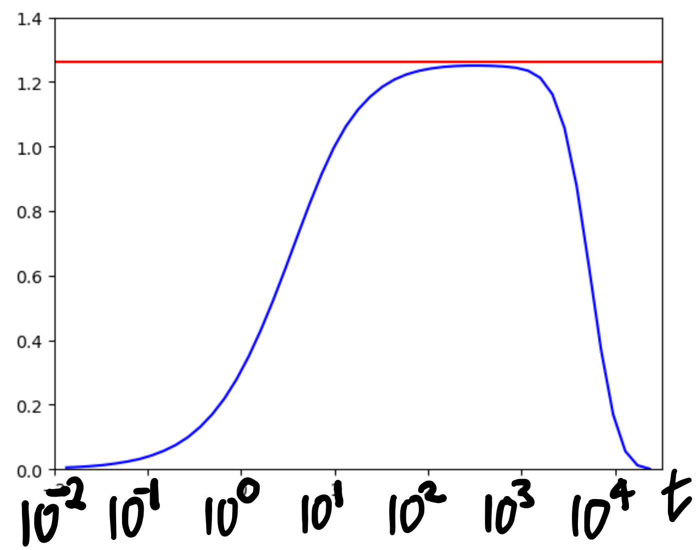
$\dim(t\cdot C)$



Koch curve, K, with 12,529 points



$\dim(t\cdot K)$

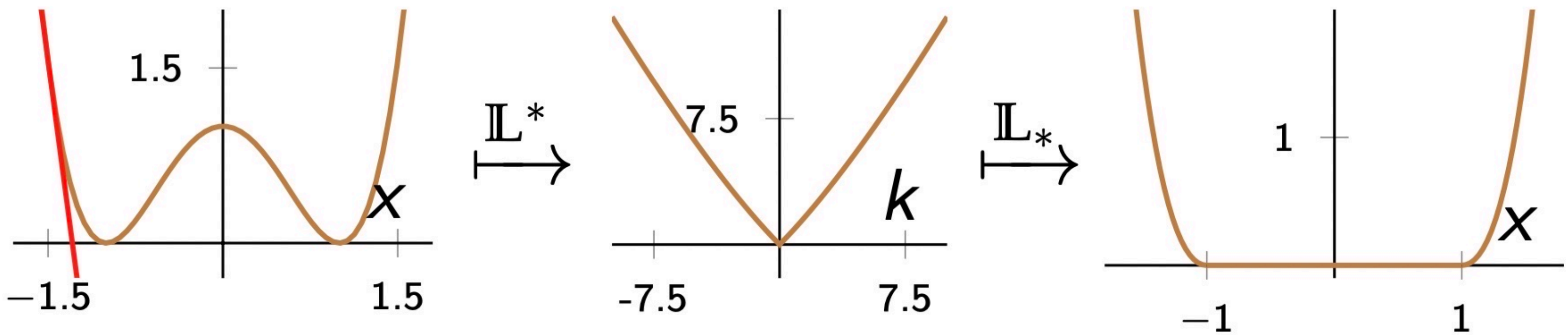


LEGENDRE - FENCHEL TRANSFORM I

V real vector space, $V^\#$ its dual, $\overline{\mathbb{R}} = [-\infty, +\infty]$

$$\mathbb{L}^*: \mathrm{Fun}(V, \overline{\mathbb{R}}) \rightleftarrows \mathrm{Fun}(V^\#, \overline{\mathbb{R}}) : \mathbb{L}_*$$

$$\mathbb{L}^*(f)(k) = \sup_{x\in V}\{\langle k,x\rangle - f(x)\}, \quad \mathbb{L}_*(g)(x) = \sup_{k\in V^\#}\{\langle k,x\rangle - g(k)\}$$



The image is always a (lower semicontinuous) convex function.

The composites $\mathbb{L}_*\mathbb{L}^*$ & $\mathbb{L}^*\mathbb{L}_*$ are "convex hull" operators.

We get an isomorphism between the sets of convex functions.

$$\mathrm{Cvx}(V, \overline{\mathbb{R}}) \cong \mathrm{Cvx}(V^\#, \overline{\mathbb{R}})$$

LEGENDRE-FENCHEL TRANSFORM II

$\bar{\mathbb{R}} = ((\,[-\infty,+\infty],\geq),+,0)$ An $\bar{\mathbb{R}}$-category can have negative distances.

Consider V & $V^\#$ as discrete $\bar{\mathbb{R}}$-categories:

$$V(x,x') = \begin{cases} 0 & \text{if } x = x' \\ +\infty & \text{if } x \neq x' \end{cases}$$

Pairing:

$$\amalg : V \otimes V^\# \to \bar{\mathbb{R}}; \quad (x,k) \mapsto \langle k,x\rangle = k(x)$$

Get $\bar{\mathbb{R}}$-adjunction

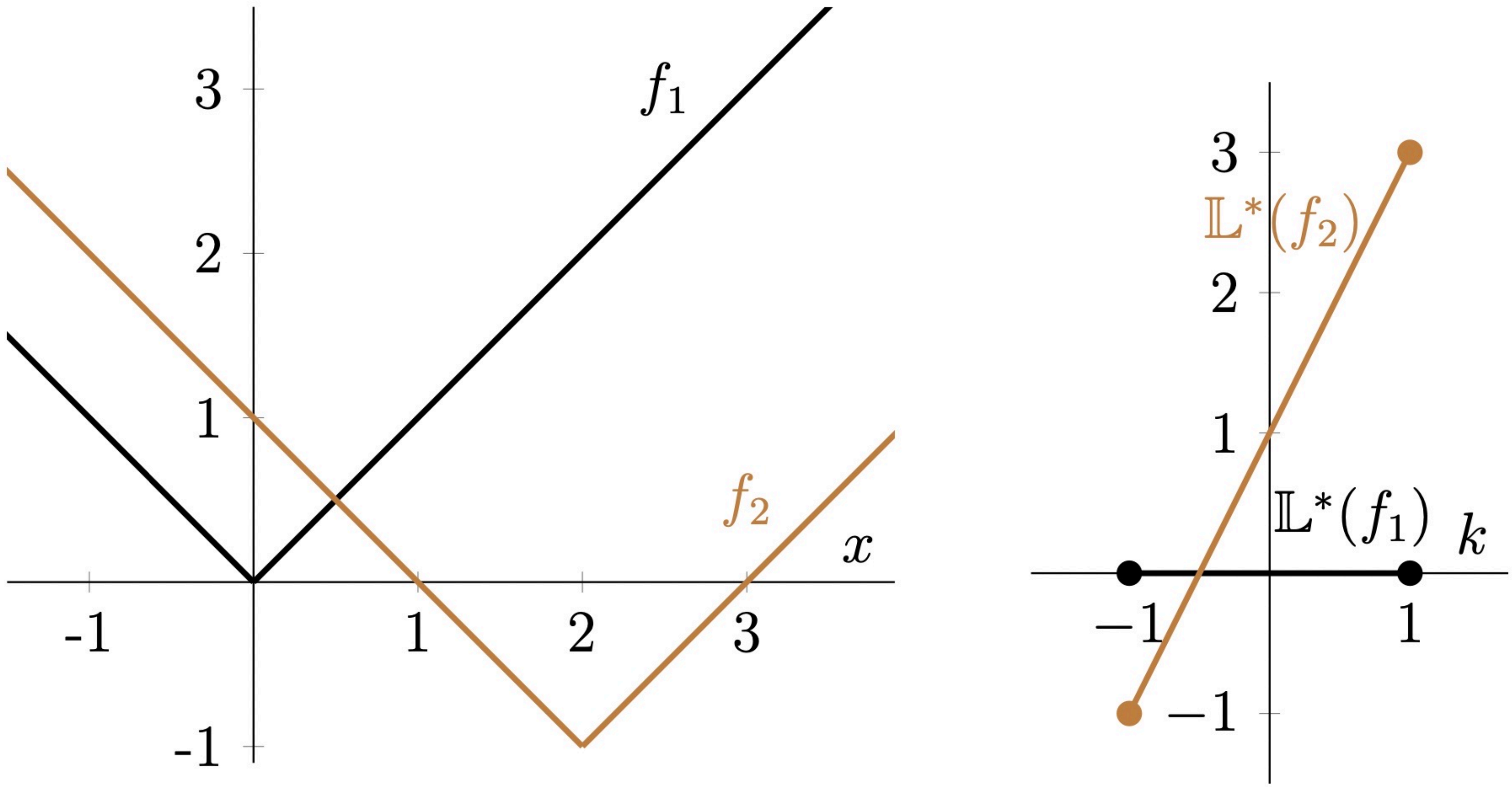
$$[\![V^{op},\bar{\mathbb{R}}]\!] \underset{\amalg_*}{\overset{\amalg^*}{\underset{\perp}{\rightleftarrows}}} [\![V^\#,\bar{\mathbb{R}}]\!]^{op}$$ ⟵ is the LFT

Nucleus gives an $\bar{\mathbb{R}}$-isometry:

$$\mathrm{Cvx}(V,\bar{\mathbb{R}}) \cong \mathrm{Cvx}(V^\#,\bar{\mathbb{R}})^{op}$$ Toland-Singer Duality.

Much of Rockafellar's "Convex Analysis" is expressible as $\bar{\mathbb{R}}$-category theory.

Hope: new ways to organise and implement optimization algorithms.



f_1
f_2
x
3
2
1
-1
-1
1
2
3
$\mathbb{L}^*(f_2)$
$\mathbb{L}^*(f_1)$
k
3
2
1
-1
-1
1