Post-groups and post-Lie algebras in differential geometry

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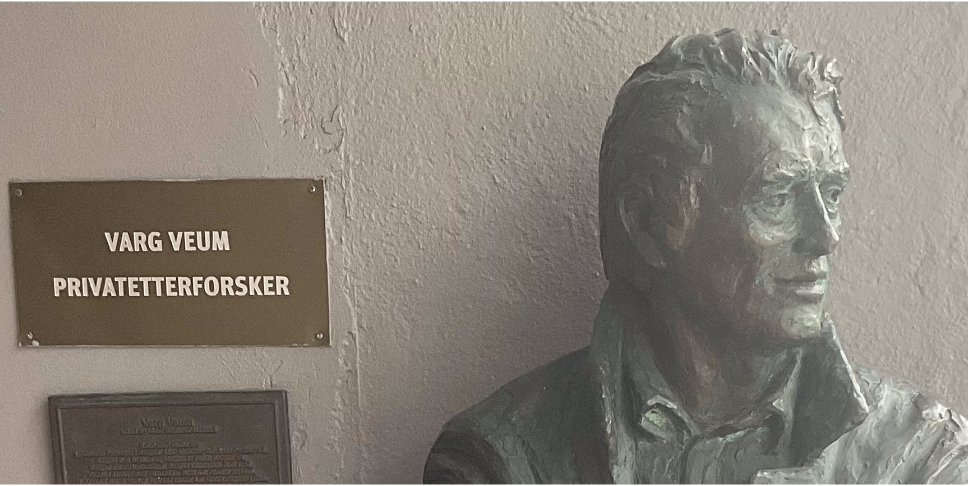
1. Affine connections
2. Pre-Lie and post-Lie
3. Post-Lie approach to connections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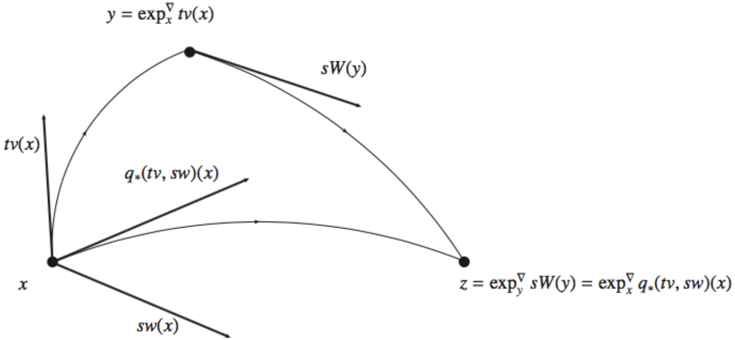
Based on joint works with

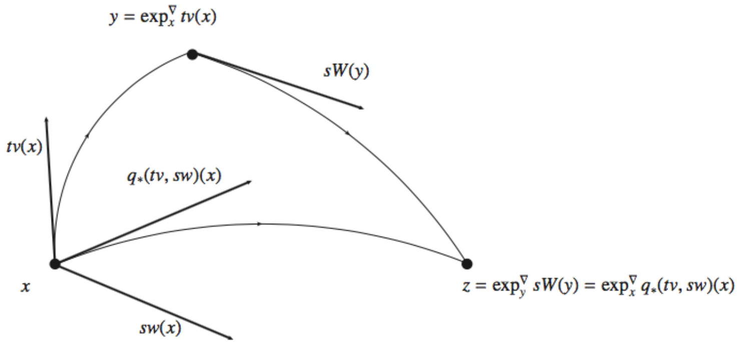
- **Mahdi Jasim Hasan Al-Kaabi** (Mustansiriyah University, Baghdad, Iraq),
- **Kurusch Ebrahimi-Fard** (NTNU, Trondheim),
- **Hans Z. Munthe-Kaas** (UiB, Bergen),

(arXiv:2205.04381 and 2306.08284)

- **Yuanyuan Zhang** (Henan University, China).



GAVRILOV'S DOUBLE EXPONENTIAL



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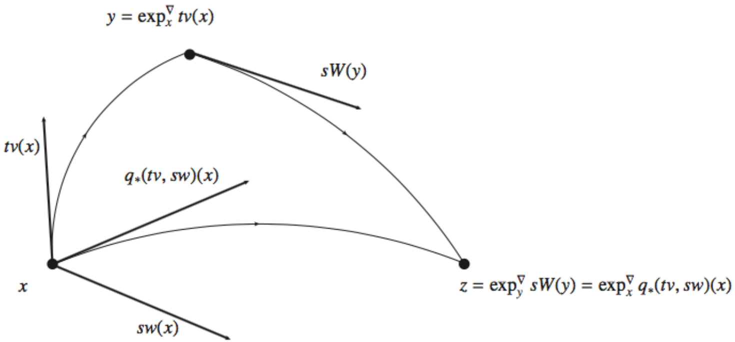
$$q_*(tv, sw) = \beta^{-1}\Big(\mathrm{BCH}\Big\{\beta(tv), \beta\big(s\lambda(tv, w)\big)\Big\}\Big)$$



GAVRILOV'S DOUBLE EXPONENTIAL

$$q_*(tv, sw) = \beta^{-1}\Big(\mathrm{BCH}\Big\{\beta(tv), \beta\big(s\lambda(tv, w)\big)\Big\}\Big)$$

is a **tensorial quantity**. Its evaluation at $x \in \mathcal{M}$ only depends on $v(x)$ and $w(x)$.

Affine connections

- Let $\mathcal{M}$ be a C^∞ manifold, let

$$\mathcal{X}\mathcal{M} = \{\text{vector fields on } \mathcal{M}\}.$$

- ∇ affine connection $\Rightarrow$ covariant derivative operator

$$\nabla_{fX}Y = f\nabla_X Y, \qquad \nabla_X(fY) = f\nabla_X Y + (X.f)Y$$

for any $f \in C^\infty(\mathcal{M})$ and $X, Y \in \mathcal{X}\mathcal{M}$.

- Notation:

$$X \triangleright Y := \nabla_X Y.$$

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- Notation:

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- Lie-Jacobi bracket on $\mathcal{X}\mathcal{M}$:

$$[X, Y]f := X.(Y.f) - Y.(X.f).$$

- $(\mathcal{X}\mathcal{M}, \rhd, [-,-])$ is a **framed Lie algebra** (terminology due to A. V. Gavrilov).

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- $(\mathcal{XM}, \rhd, [\![-,-]\!])$ is a **framed Lie algebra** (terminology due to A. V. Gavrilov).
- It is also a $C^\infty(\mathcal{M})$-module.
- No compatibility between $\rhd$ and $[\![-,-]\!]$ in the definition of a framed Lie algebra.
- Two rather involved relations between $\rhd$ and $[\![-,-]\!]$ in $\mathcal{XM}$: the **Bianchi identities**.

- Torsion:
$$t(X,Y) := X \rhd Y - Y \rhd X - [X,Y].$$
- Curvature:
$$r(X,Y)(Z) = R(X,Y,Z) := X \rhd (Y \rhd Z) - Y \rhd (X \rhd Z) - [X,Y] \rhd Z.$$

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- The curvature in $C^\infty(\mathcal{M})$-linear w.r.t. its three arguments.
- t and R are the building blocks of the **special polynomials**, like
$$R\Big((X \rhd R)(Y,Z,T), U, t\big(V, R(W,A,B)\big)\Big)$$

Bianchi identities

- **First Bianchi identity:**

$$\oint_{XYZ} R(X,Y,Z) = \oint_{XYZ} (X \rhd t)(Y,Z) - \oint_{XYZ} t\big(X, t(Y,Z)\big)$$

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- **Second Bianchi identity:**

$$\oint_{XYZ} (X \rhd R)(Y,Z,W) = \oint_{XYZ} R\big(X, t(Y,Z), W\big)$$

with $(X \rhd R)(Y,Z,W) := X \rhd R(Y,Z,W) - R(X \rhd Y, Z, W) - R(Y, X \rhd W, Z) - R(Y, Z, X \rhd W)$.

The flat torsion-free case (example: $\mathcal{M} = \mathbb{R}^n$)

.

- If $t(X, Y) = 0$ and $r(X, Y) = 0$ for any $X, Y \in \mathcal{X}\mathcal{M}$, the **left pre-Lie identity** holds:

$$X \rhd (Y \rhd Z) - (X \rhd Y) \rhd Z = Y \rhd (X \rhd Z) - (Y \rhd X) \rhd Z$$

(Vinberg 1963, Gerstenhaber 1963, Cayley 1857).

The flat constant torsion case (Examples: homogeneous spaces)

- We suppose
  - Flatness condition: $r(X,Y)=0$ for any $X,Y\in\mathcal{XM}$,
  - Constant torsion condition:

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- $(\mathcal{X}\mathcal{M}, \rhd, [\![-,-]\!])$ is a **post-Lie algebra** (B. Vallette 2007) and the differential operators $(\mathcal{D}\mathcal{M}, \rhd, \cdot)$ form a **D-algebra** (H. Z. Munthe-Kaas and W. Wright 2008).

The post-Lie algebra axioms

- $a \triangleright [b, c] = [a \triangleright b, c] + [b, a \triangleright c]$,
- $[a, b] \triangleright c = a \triangleright (b \triangleright c) - (a \triangleright b) \triangleright c - b \triangleright (a \triangleright c) + (b \triangleright a) \triangleright c$.

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- A post-Lie algebra with vanishing bracket is a pre-Lie algebra.

Take-home message

The post-Lie formalism is still relevant for any affine connection on $\mathcal{M}$.

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- **Beware:** $[-,-] \neq \mathbf{[}-,-\mathbf{]}$!
- Both $\widetilde{\mathfrak{g}}$ and $\mathfrak{g}$ are post-Lie algebras. Extending the scalars to $s\mathbb{R}[s]$ makes them positively graded post-Lie algebras.

The Grossman-Larson bracket

In any post-Lie algebra $\mathfrak{g}$,

$$[\![X, Y]\!] =: [X, Y] + X \rhd Y - Y \rhd X$$

defines a Lie bracket. It is **not** $C^\infty(\mathcal{M})$-linear, contrarily to $[-, -]$.

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Opposite post-Lie algebra $\mathfrak{g}^{\text{op}} := (\mathfrak{g}, -[-,-], \blacktriangleright)$ with

$$X \blacktriangleright Y := X \triangleright Y + [X, Y],$$

sharing the **same** Grossman-Larson bracket.

Torsion and curvature revisited

For any $X, Y, Z \in \mathcal{V} = \mathcal{X}\mathcal{M}$,

- $t(X,Y) = [\![X,Y]\!] - [X,Y] - \mathbf{[}X,Y\mathbf{]}$,
- $R(X,Y,Z) = \big([\![X,Y]\!] - \mathbf{[}X,Y\mathbf{]}\big) \rhd Z$.

Post-Lie Magnus expansion

$$
\begin{tikzcd}
\widehat{\mathfrak{g}} \arrow[rr, "\exp^{\cdot}"] \arrow[d, shift right, "\chi"'] & & G \subset \widehat{\mathcal{U}(\mathfrak{g})} \\
\widehat{\mathfrak{g}} \arrow[u, shift right, "\theta"'] \arrow[urr, "\exp^{*}"'] & &
\end{tikzcd}
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- Post-Lie Magnus expansion $\chi : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$

$$\chi(X) := \log^{*} \exp^{\cdot}(X).$$

- Inverse post-Lie Magnus expansion:

$$\theta(X) := \chi^{-1}(X) = \log^{\cdot} \exp^{*}(X).$$

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- $\chi(X) =$
$X - \frac{1}{2}X \triangleright X + \frac{1}{2}X \triangleright (X \triangleright X) + \frac{1}{4}(X \triangleright X) \triangleright X + \frac{1}{2}[X \triangleright X, X] + \cdots$
- $\theta(X) = X + \frac{1}{2}X \triangleright X + \frac{1}{6}X \triangleright (X \triangleright X) + \frac{1}{2}[X, X \triangleright X] + \cdots$

Gavrilov's K-map

- V vector space, $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ its tensor algebra,
- $\rhd : V \times V \to V$ bilinear binary product (no other property assumed).

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$$K(x_1\otimes\cdots\otimes x_n)=x_1\otimes K(x_2\otimes\cdots\otimes x_n)$$
$$-\sum_{j=2}^{n} K\big(x_2\otimes\cdots\otimes(x_1\rhd x_j)\otimes\cdots\otimes x_n\big).$$

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$$K(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes K(x_2 \otimes \cdots \otimes x_n)$$
$$- \sum_{j=2}^{n} K\big(x_2 \otimes \cdots \otimes (x_1 \rhd x_j) \otimes \cdots \otimes x_n\big).$$

- $K(x \otimes y) = x \otimes y - x \rhd y$,

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- $K(x \otimes y \otimes z) = x \otimes y \otimes z - x \otimes (y \rhd z) - (x \rhd y) \otimes z - y \otimes (x \rhd z)$
$+ (x \rhd y) \rhd z + y \rhd (x \rhd z)$.

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$$y_B := y_{b_1} \rhd \Big(y_{b_2} \rhd \big(\cdots \rhd (y_{b_{\ell-1}} \rhd y_{b_\ell}) \ldots \big) \Big)$$

if the block $B \subset \{1, \ldots, n\}$ is of cardinality ℓ, and where the blocks are ordered wrt their maximum.

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- $K^{-1}(x \otimes y) = x \otimes y + x \rhd y$,
- $K^{-1}(x \otimes y \otimes z) =$
  $x \otimes y \otimes z + (x \rhd y) \otimes z + y \otimes (x \rhd z) + x \rhd (y \otimes z) + x \rhd (y \rhd z)$.

Two commuting diagrams

$$\begin{tikzcd}
(T_{\mathbb{R}}(\mathcal{V}),*) \arrow[rr, "K", "\sim"'] \arrow[d, "\pi"] & & (T_{\mathbb{R}}(\mathcal{V}),\cdot) \arrow[d, "p"] \\
(T_{C^\infty(\mathcal{M})}(\mathcal{V}),*) \arrow[drr, "\rho"] & & \mathcal{U}(\mathcal{V}) \arrow[d, "\widetilde{\pi}"] \\
& & \mathcal{DM}
\end{tikzcd}
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\begin{tikzcd}
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Here ρ stands for the **higher-order covariant derivative map**.

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K is a **Hopf algebra isomorphism** (A. V. Gavrilov, 2012).

Gavrilov's β map in terms of post-Lie Magnus expansion

$$
\begin{tikzcd}
 & \overline{\widetilde{\mathfrak{g}}} \arrow[r, "\widetilde{\chi}", "\sim"'] \arrow[d, two heads, "\pi"] & (\overline{\widetilde{\mathfrak{g}}}, [\![. \, , .]\!]) \arrow[r, "K", "\sim"'] \arrow[d, two heads, "\pi"] & (\overline{\widetilde{\mathfrak{g}}}, [. \, , .]) \arrow[dd, two heads, "p"] \\
 & \overline{\mathfrak{g}} \arrow[r, "\chi", "\sim"'] & (\overline{\mathfrak{g}}, [\![. \, , .]\!]) \arrow[dr, "\rho"] & \\
\overline{\mathcal{V}} \arrow[uur, hook, "\widetilde{\iota}"] \arrow[ur, hook, "\iota"'] \arrow[rrr, "\beta", "\sim"'] & & & \overline{\mathcal{V}}
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$$\beta = p \circ K \circ \widetilde{\chi} \circ \widetilde{\iota}$$

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$$\exp \beta(X) = p \circ K(e^{\cdot X}) = \rho \circ \pi(e^{\cdot X}).$$

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- $u \rhd (v.w) = \sum_{(u)} (u_1 \rhd v).(u_2 \rhd w)$,
- $u \rhd (v \rhd w) = (u * v) \rhd w$

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  with $u * v := \sum_{(u)} u_1.(u_2 \rhd v)$,

Post-Hopf algebras (Y. Li, Y. Sheng, R. Tang, 2022)

The universal enveloping algebra of a post-Lie algebra is a D-algebra, and more precisely a **post-Hopf algebra**, namely a (cocommutative) Hopf algebra $(H, ., \Delta, u, \varepsilon, S)$ together with a linear map $\triangleright : H \otimes H \to H$ such that

- $u \triangleright (v.w) = \sum_{(u)} (u_1 \triangleright v).(u_2 \triangleright w)$,
- $u \triangleright (v \triangleright w) = (u * v) \triangleright w$
  with $u * v := \sum_{(u)} u_1.(u_2 \triangleright v)$,
- The map $L^{\triangleright} : H \to \mathcal{L}(H, H)$ defined by $L^{\triangleright}_u := u \triangleright -$ admits an inverse $\beta^{\triangleright}$ for the convolution product.

The following facts hold:

- The tuple $\widetilde{H} := (H, *, \Delta, u, \varepsilon, S_*)$ is a Hopf algebra with antipode given by $S_*(u) := \sum_{(u)} \beta^{\triangleright}_{u_1}\big(S_.(u_2)\big)$,
- The product . can be recovered from $*$ by

$$u.v = \sum_{(u)} u_1 * \big(S_*(u_2) \triangleright v\big).$$

A. V. Gavrilov's initial value problem

For any x, y in a post-Lie algebra $\mathfrak{g}$, the quantity

$$\lambda(tx, y) := S_*\big(\exp^{\cdot}(tx)\big) \rhd y = \exp^*\big(-\chi(tx)\big) \rhd y$$

is the solution of the differential equation

$$\dot{\lambda}(tx, y) = -\lambda(tx, x) \rhd \lambda(tx, y)$$

with initial value $\lambda(0, y) = y$.

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is the solution of the differential equation

$$\dot{\lambda}(tx, y) = -\lambda(tx, x) \rhd \lambda(tx, y)$$

with initial value $\lambda(0, y) = y$. In particular, $\alpha(tx) := \lambda(tx, x)$ is the solution of the differential equation

$$\dot{\alpha}(tx) = -\alpha(tx) \rhd \alpha(tx)$$

with initial value $\alpha(0x) = x$.

$\lambda(tx, y)$ is the **parallel transport at time t of y in the direction of x.**

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This object is purely magmatic:

$$\lambda(tx,y) = y - tx \rhd y + \frac{t^2}{2}\big(x \rhd (x \rhd y) + (x \rhd x) \rhd y\big)$$

$$-\frac{t^3}{6}\Big(x \rhd \big(x \rhd (x \rhd y)\big) + x \rhd \big((x \rhd x) \rhd y\big) + \big(x \rhd (x \rhd x)\big) \rhd y$$

$$+\big((x \rhd x) \rhd x\big) \rhd y + 2(x \rhd x) \rhd (x \rhd y)\Big) + \cdots .$$

This matches the differential geometric notion when $x, y \in \mathcal{X}\mathcal{M}$.

Post-groups (C. Bai, L. Guo, Y. Sheng, R. Tang, 2023)

A **post-group** is a group $(G, .)$ endowed with $\triangleright : G \times G \to G$ such that

- $L_A^{\triangleright} : a \triangleright -$ is a group automorphism for any $a \in G$,
- $a \triangleright (b \triangleright c) = (a * b) \triangleright c$, with

$$a * b := a.(a \triangleright b).$$

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It turns out that

- $(G, *)$ is the **Grossman-Larson group**, sharing the same unit with $(G,.)$,
- $(G, *)$ acts on $(G,.)$ by automorphisms.

- The **opposite post-group** is given by $(G, \odot, \blacktriangleright)$ with $a \odot b := b.a$ and $a \blacktriangleright b := a.(a \triangleright b).a^{-1}$.

- Both share the same Grossman-Larson group: $a * b = a.(a \triangleright b) = a \odot (a \blacktriangleright b)$.

- The Lie algebra of a **post-Lie group** is a post-Lie algebra [BGST2023]:

$$X \rhd Y = \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\exp(tX) \rhd \exp(sY).$$

- The post-Lie algebra of the opposite post-Lie group is the opposite post-Lie algebra.

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- The post-Lie algebra of the opposite post-Lie group is the opposite post-Lie algebra.

- **Theorem** (BGST2023): The notion of post-group is equivalent to
    - Guarnieri-Vendramin's **skew-braces** (2017),
    - Lu-Yan-Zhu's **braided groups** (2000).

Braided groups (J.-H. Lu, M. Yan, Y.-C. Zhu, 2000)

A **braided group** is a triple (G, m, σ) where (G, m) is a group and

$$\sigma : G \times G \xrightarrow{\sim} G \times G$$

such that

- $\sigma \circ (m \times m) = (m \times m) \circ \widetilde{\sigma}$, with

$$\widetilde{\sigma} := \sigma_{23} \circ (\sigma \times \sigma) \circ \sigma_{23} : G^4 \xrightarrow{\sim} G^4,$$

- $m \circ \sigma = m$.

- σ verifies the **braid equation**

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- Introducing the notation $\sigma(x,y)=(x\rightharpoonup y,\, x\leftharpoonup y)$, both maps $\rightharpoonup$ and $\leftharpoonup$ are respectively a left and a right action of G on itself.

[LYZ2000].

Theorem [BGST2023]

- For any post-group $(G, ., \triangleright)$, the Grossman-Larson group $(G, *)$ is a braided group, with σ given by

$$\sigma(g,h) = \Big(g \triangleright h, (g \triangleright h)^{*-1} * g * h\Big).$$

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- Conversely, any braided group $(G, *, \sigma)$ with

$$\sigma(g,h) = (g \rightharpoonup h, g \leftharpoonup h)$$

gives rise to a post-group $(G, ., \triangleright)$ whose GL product is $*$. Explicitly,

$$g \triangleright h := g \rightharpoonup h, \qquad g.h := g * (g^{*-1} \rightharpoonup h).$$

Opposite post-group revisited

Proposition [AEM2023]: The braiding corresponding to the opposite post-group $(G, \odot, \blacktriangleright)$ is σ^{-1}.

Opposite post-group revisited

Proposition [AEM2023]: The braiding corresponding to the opposite post-group $(G, \odot, \blacktriangleright)$ is σ^{-1}.

As a consequence, **pre-groups** are the same as **symmetric braided groups**.

Left-regular diagonal magmas and the $\mathcal{K}$ map

Definition [AEM2023]: A **left-regular diagonal magma** is a magma $(M, \triangleright)$ such that

- The maps $L_a^{\triangleright} = a \triangleright -$ are bijective for any $a \in M$,
- The map $\Lambda : M \to M$ given by $\Lambda(a) := (L_a^{\triangleright})^{-1}(a)$ is bijective.

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Proposition [AEM2023]: For any left-regular diagonal magma M,

- the free group F_M is a post-group, fulfilling the universal property below

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, hook] & (G, ., \triangleright) \\ F_M \arrow[ur, dashed, "\widetilde{\varphi}"'] & \end{tikzcd}$$

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- there is a unique explicit group isomorphism $\mathcal{J} : (F_M, .) \to (F_M, *)$ extending Id_M. Its inverse $\mathcal{K}$ is a post-group analogue of Gavrilov's K-map.

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Thank you very much!