A Framework for Universality in Physics, Computer Science, and Beyond

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References

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[La69] F. W. Lawvere,
Diagonal arguments and cartesian closed categories.
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Motivation

Motivating examples

Primary:

- universal Turing machines
- universal spin models [De16]

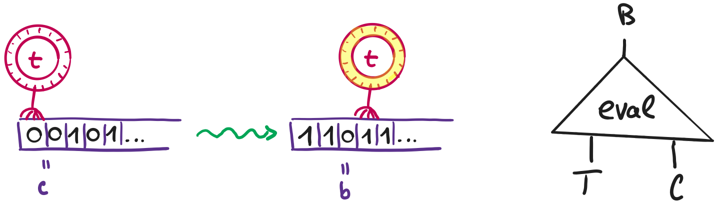
Others:

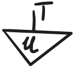
- NP-completeness
- dense subsets
- universal approximation by neural networks
- generating sets (universal gate set, . . .)
- universal graphs
- xenobots and chemputers
- universal explanations (emergent properties) and universals in metaphysics

Goals of the framework

- Unified language
- Examples of universality
- Knowledge-organization
- General theory of universality
  - trivial vs. non-trivial universality
  - necessary conditions for universality
  - Fixed-point theorem + relation to undecidability

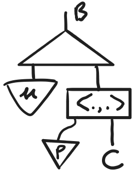
Universal Turing machine





is universal ≈ u can emulate any $t \in T$:

$\exists\ p = \boxed{0\,|\,0\,|\,1\,|\,1\,|\,1}$:



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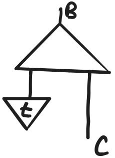
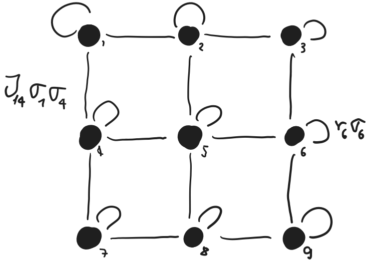


Spin system

- Spin d.o.f. Σ for each vertex
- A hypergraph, edges $\sim$ local interactions
- Hamiltonian $\Sigma^V \to \mathbb{R}$ as a sum of local coupling terms

2D Ising spin model with fields has $\Sigma = \mathbb{Z}_2$ and interaction lattice:

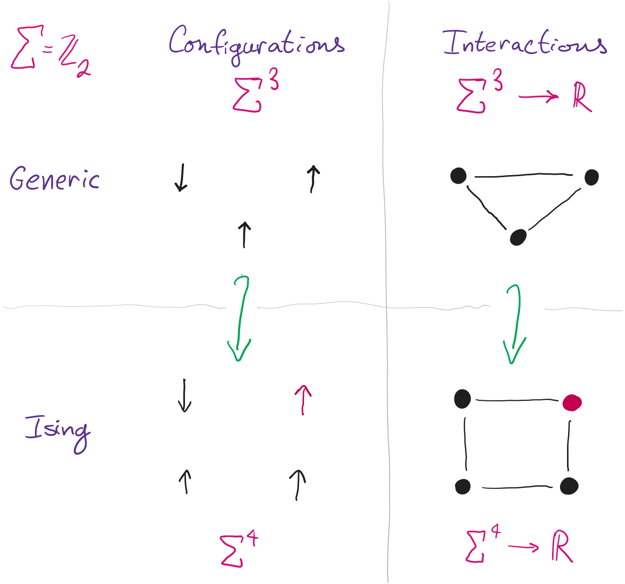


$$\sigma_i \in \Sigma$$

$$H = \sum_{i,j} J_{ij}\,\sigma_i\sigma_j + \sum_i r_i\,\sigma_i$$

Spin system simulation

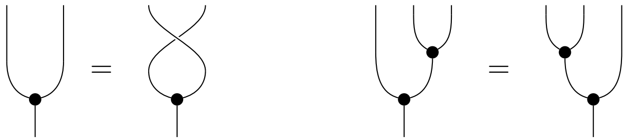
Every spin system can be simulated on a 2D Ising one [De16].

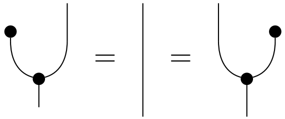


The set-up (simulators)

Ambient category

$\mathcal{A}$ is **gs-monoidal**, an SMC with $A \to A \otimes A$ and $A \to I$ such that

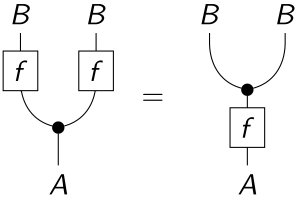




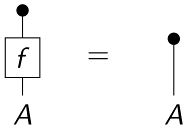
Key examples: Rel, Set, Rel_{poly}, Comp($\mathbb{N}$), Poly($\mathbb{N}$)

Deterministic

functional morphism:



total morphism:

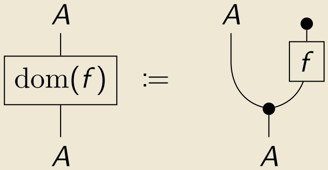


Functional + total = **deterministic**

Domain

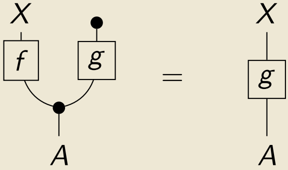
Definition ([Fr22])

The **domain** of $f : A \to X$ is



Definition

f agrees with g on the domain of g, denoted $f \sqsupseteq g$, if



Target–context category

Definition

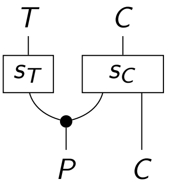
A **target–context category** $(\mathcal{A}, T, C, \gtrdot)$ is a gs-monoidal category $\mathcal{A}$ with distinguished objects T and C, and preorders $\gtrdot$ on every $\mathcal{A}(A, T \otimes C)$, such that:

- $f \sqsupseteq f$
- $f \sqsupseteq g \quad \Longrightarrow \quad f \gtrdot g$
- $f \gtrdot g \quad \Longrightarrow \quad f \circ h \gtrdot g \circ h$

for all $f, g\colon A \to T \otimes C$ and $h\colon Z \to A$.

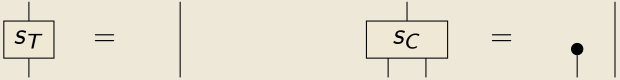
$\mathcal{A}$	**ambient cat.**	$\mathrm{Comp}(\mathbb{N})$	$\mathrm{Rel}_{\mathrm{poly}}$
A	**object**	$\mathbb{N}^n$ for $n \in \mathbb{N}$	“sized” sets
f	**morphism**	computable fun.	bounded relations
T	**targets**	Turing machines	spin systems
C	**contexts**	input strings	spin configurations
$\gtrdot$	**ambient rel.**	computation	energy condition

A simulator

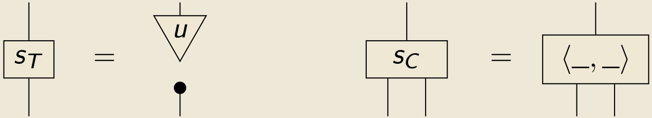


$P \in \mathcal{A}$	**programs**
$s_T : P \to T$	**compiler**
$s_C : P \otimes C \to C$	**context reduction**

Example (trivial simulator)



Example (singleton simulator for TM)

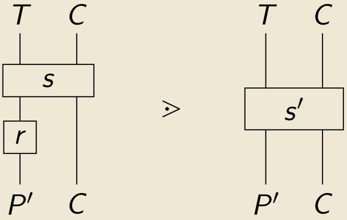


Universality

Reductions and universality

Definition

A **lax reduction** $r^*\colon s \to s'$ from simulator s to s' is a functional $r\colon P' \to P$ such that



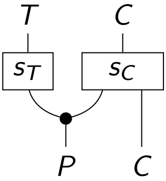
Definition

Simulator s is **universal** if there is a lax reduction $s \to \mathrm{id}$ to the trivial simulator.

Examples (universal simulators)

- Trivial simulator
- Singleton simulator for a universal TM
- 2D Ising spin model with fields
- NP-complete language
- Dense subset (e.g. $T = \mathbb{R} \times \mathbb{R}_+$, $P = \mathbb{Q} \times \mathbb{R}_+$)
- A generating set (T = tuples, C = formulas)
- Weak limit (T = cones, $\gtrdot$ = cone factorization, $P = C = I$)
- Cofinal subset P of a poset $(T, \gtrdot)$.

No-go theorem



T	spin systems
$s_T\colon P \hookrightarrow T$	2D Ising
$s_C\colon P \otimes C \to C$	config. embedding

Theorem

For a "suitably $\gtrdot$-monotone" function $\varphi\colon T \to \mathbb{R}$, we have

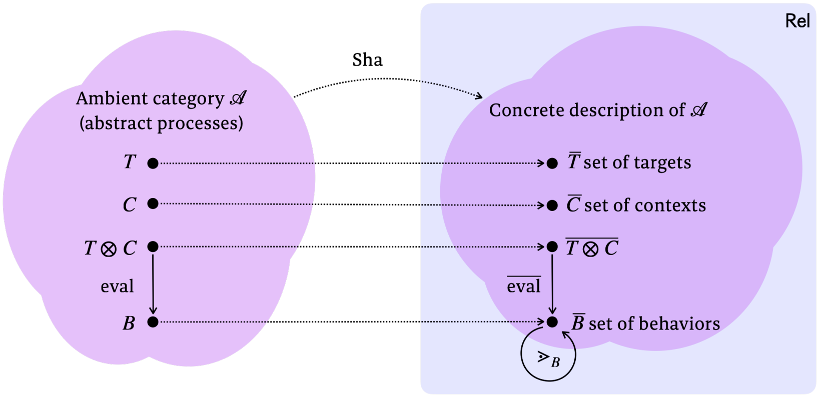
$$s \gtrdot \mathrm{id}_{T\otimes C} \quad \Longrightarrow \quad \sup \varphi\big(\mathrm{im}(s_T)\big) \geq \sup \varphi(T).$$

For spin systems, $\varphi = |\mathrm{spec}|$ works, and RHS is ∞, so

a universal spin model cannot be finite.

Relation to undecidability

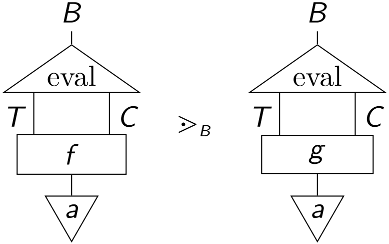
Intrinsic behavior structure



Intrinsic behavior structure

T	**targets**	TMs	spin systems
C	**contexts**	inputs	spin configurations
B	**behaviors**	outputs	energies $+ \ldots$
$\gtrdot_B$	preorder	$=$	$\ldots$
eval	$T \otimes C \to B$	evaluation	measurement

$f \gtrdot g$:

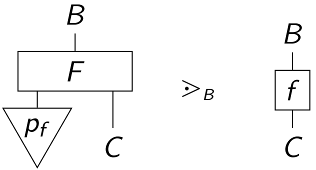


for all $a \in \mathcal{A}_{\mathrm{det}}(I, A)$, such that RHS is defined.

Unreachability from universality

$F\colon P \otimes C \to B$ is a **complete parametrization** (CP) if for every f

$\exists\, p_f \in \mathcal{A}_{\mathrm{det}}(I, P)$:



A simulator s has **unreachability** if $\mathrm{eval} \circ s$ is not a CP.

Theorem (Fixed Point Theorem à la [La69])

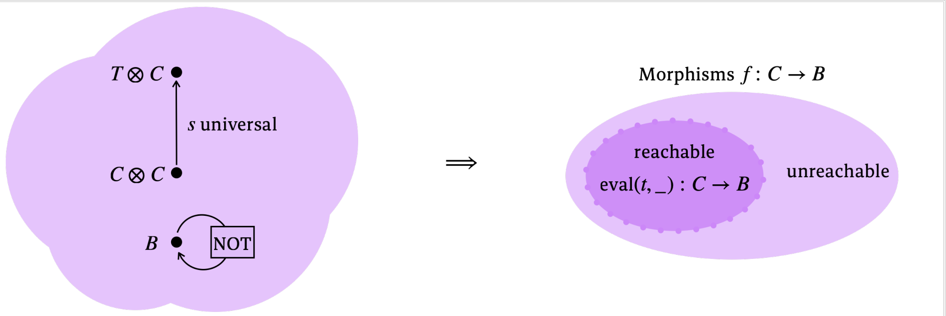
If $F\colon C \times C \to B$ is a CP, then every $g\colon B \to B$ has a (quasi) fixed point.

Lemma

If eval is a CP and s is universal, then $\mathrm{eval} \circ s$ is a CP.

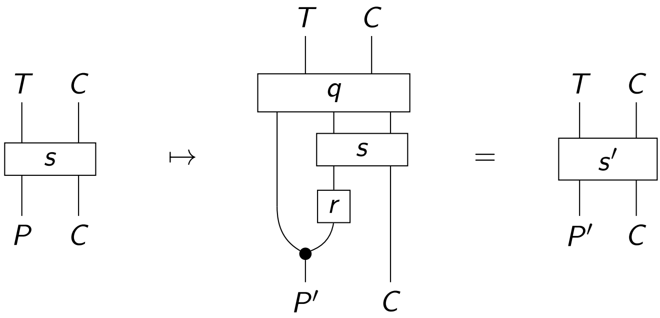
universal s + fixed-point-free $g \implies$ unreachability of id

Unreachability from universality



Hierarchy of universal simulators

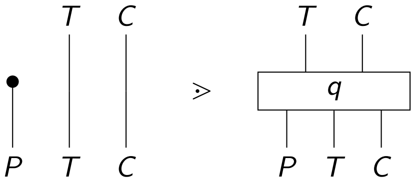
Simulator morphisms

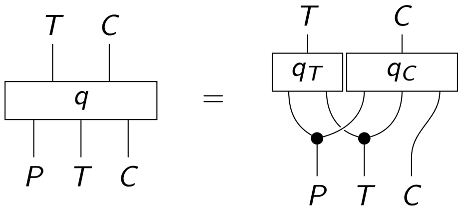


r is deterministic ensures sequential composition

We also require that (s' is universal) $\implies$ (s is universal).

Processings





Parsimony of simulators

Definition

s' is a **more parsimonious** simulator than s, written $s' \geq s$, if there exists a morphism $s \to s'$.

Theorem

The singleton simulator s_u for a universal TM is strictly more parsimonious than the trivial simulator.

- $s_u \geq \mathrm{id}$ constructs right-inverse to the reduction.
- $s_u \not\leq \mathrm{id}$ because $\exists\, t, t' \colon I \to T$ such that
  - t cannot simulate t' and
  - they both compile to $u \in T$

Summary

- Abstract notion of universality with several instances
- Necessary conditions for universality
- Morphisms of simulators $\to$ non-trivial universality
- Fixed Point Theorem and unreachability
- target–context functors & simulator categories